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On exponential corrections to the $1/N$ expansion in two-dimensional Yang Mills

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Abstract: We compute e^{-AN} corrections to the Gross-Taylor $1/N$ expansion of the partition function of two-dimensional SU(N) and U(N) Yang Mills theory. We find a very similar structure of mixing between holomorphic and anti-holomorphic sectors as that described by Vafa for the $1/N$ expansion. Some of the non-perturbative terms are suggestive of D-strings wrapping the T^2 of the 2dYM but blowing up into a fuzzy geometry by the Myers effect in the directions transverse to the T^2.

Keywords: Field Theories in Lower Dimensions, 1/N Expansion, Topological Strings.

Contents

1. Introduction

The exact partition function of 2dYM on a torus T^2 is known and expressed simply as a sum over representations of SU(N) [1].

$$Z = \sum_Y e^{-\frac{Ag_{YM}^2}{2}C_2(Y)} \tag{1.1}$$

Gross-Taylor [2, 3] have described the perturbative $1/N$ expansion of the 2dYM partition function and interpreted it in terms of maps of elementary string worldheets to the T^2. There is a free fermion description of this system [4, 5]. A topological string theory was given in [6, 7]. Non-perturbative aspects were studied in [4] and subsequently [8]. A different approach to non-perturbative aspects was pursued in [9, 10]. Recent work of [11, 12] has renewed interest in the structure of the non-perturbative expansion.

The Gross-Taylor approach first constructs a "chiral" perturbative $1/N$ expansion

$$Z_{\text{chir}}^{\text{pert}} = \sum_{n=0}^{\infty}\sum_R e^{-AC_2(R)}\,. \tag{1.2}$$

Here the Casimirs of R are computed under the assumption that R are Young diagrams with n boxes, where n is small compared to N. If small Young diagrams contribute to the perturbative expansion so do their conjugates which have columns of length close to N, as well as composites which have a few long columns and a few short columns. Taking into account these composite Young Diagrams gives the complete $1/N$ expansion which is the non-chiral partition function

$$Z_{\rm nchir}^{\rm pert} = \sum_{n=0}^{\infty}\sum_{\tilde n=0}^{\infty}\sum_{R,S} e^{-AC_2(R\bar S)}\,. \tag{1.3}$$

We would like to generalize these calculations to include Young Diagrams which contribute at order e^{-AN}. One class of such diagrams is given in [8]. They consider Young diagrams containing k rows of length N. k is of order 1 as N goes to infinity. Attached to this rectangular block of kN boxes, there are two small Young diagrams R_1 and R_2, with number of boxes of order 1 as N goes to infinity. R_1 is attached at the bottom left and R_2 is attached at the right. As observed in [8] we also need to consider, along the lines of [2, 3], the SU(N) duals of these Young diagrams, and subsequently form the chiral-antichiral composites.

There is a simple generalization of this kind of Young diagram which is also relevant. We can consider k rectangular blocks of length lN, where l is a positive integer of order 1 as N goes to infinity. These Young diagrams also give contributions of the type e^{-AN}, as can be seen from the formula The Casimir of (k, l, R_1, R_2) can be written as

$$\begin{aligned}\frac{C_2^{\rm SU}(k,l,R_1,R_2)}{N} &= Nkl(l+1) + n(R_1) + n(R_2) - k^2 l(l+1) + 2ln(R_2) + \\ &+ \frac{2kl\ (\ n(R_1)+n(R_2)\ )}{N} + \frac{\tilde C(R_1)+\tilde C(R_2)}{N} - \frac{n^2(R_1)}{N^2} - \frac{n^2(R_2)}{N^2}\,.\end{aligned} \tag{1.4}$$

Here $\tilde C(R) = \sum_i r_i(R)\ (\ r_i(R) - 2i + 1)$ following the notation of [2]. It is clear from this formula that for Young Diagrams with l being any number of order 1 as N goes to infinity, we get contributions to the partition function (1.1) , which go like finite powers of e^{-AN}. In section 2 we will consider a generalization where (k, l, R_1, R_2) is replaced by Young Diagrams characterized by two vectors of integers $\vec k, \vec l$ with p components as well as small Young Diagrams $R_1, R_2, \ldots, R_{p+1}$. In section 3, we consider composite Young Diagrams built from these generalized chiral Young Diagrams, together with their anti-chiral counterparts. We compute the Casimirs and their corresponding contribution to the partition function of two dimensional Yang Mills. In section 4, we discuss the mixing between chiral and anti-chiral contributions of these composite Young Diagrams, and observe that the structure describing this mixing given in [12] for the $1/N$ expansion continues to hold for the e^{-AN} terms.

In section 5, we discuss the physical interpretation of the e^{-AN} contributions. The plausible suggestion that they are related to D1-branes has been made in [8]. This is discussed further and the extra parameter l (and more generally $\vec l$) poses an interesting challenge since it leads to $e^{-ANkl(l+1)}$ which can be interpreted in terms of D-string wrappings only if the effective brane tension seen on the T^2 base space of the Yang Mills is proportional to $l(l+1)$. One striking feature of the proposal in [12] is that the string dual for Yang Mills on T^2 involves a six-dimensional Calabi-Yau geometry, which is a bundle $\mathcal{L}_m \oplus \mathcal{L}_{-m}$ over T^2. More generally the Calabi-Yau would naturally sit in a ten-dimensional background. Our attempts to explain the $l(l+1)$ rely on these extra dimensions of the geometry.

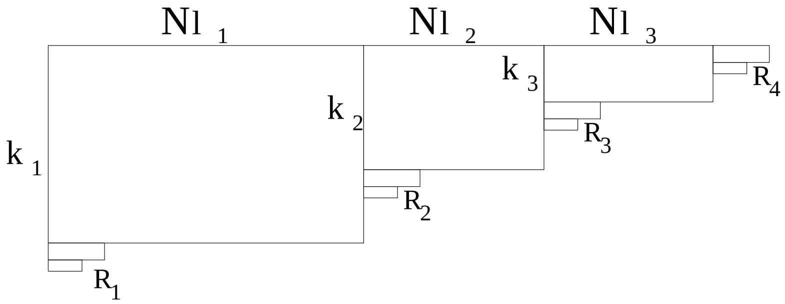

Figure 1: Picture of a large chiral Young Diagram. The positive integers k_i, l_i are or order 1 as $N \to \infty$.

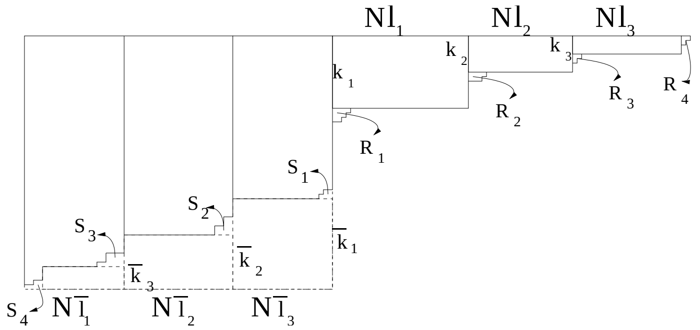

Figure 2: Picture of a large non-chiral Young Diagram. The positive integers $k_i, l_i, \bar{k}_i, \bar{l}_i$ are or order 1 as $N \to \infty$.

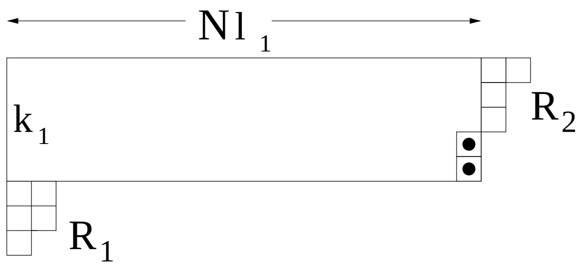

Figure 3: Picture of large chiral Young Diagram with negative row lengths for R_2. Example with $p = 1$ showing R_2 having row lengths $(2, 1, 1, -1, -1)$. The dotted boxes corresponding to the negative row entries are deleted from the large block of size $N k_1 l_1$.

2. e^{-AN} effects in $\mathrm{SU}(N)$ 2dYM and large chiral Young diagrams

The Young Diagrams of [8] involve a block with k rows of length N. We mentioned above that we can also have k rows of length lN where l is a positive integer of order 1 as $N \to \infty$. There are is further simple generalization, where we have several such blocks. These Young Diagrams with several large blocks can also have smaller, order 1 Young diagrams attached to them. Figure 1 describes these Young Diagrams. Suppose we have one such block with k_1 long rows of length $l_1 N$. To the right of it we attach another Young Diagram which has k_2 long rows of length $l_2 N$. There is a restriction $k_1 > k_2$. Further to the right we have a block with k_3 long rows of length $l_3 N$, and $k_3 > k_2$. And on up to (k_p, l_p). To this we can attach a small Young diagram (with order 1 boxes as N goes to infinity) to the bottom of the first (k_1, l_1) block. Another small Young diagram is added to the bottom of the second rectangular block, and so on up to the p'th block, and finally a small Young diagram can be attached to the right of the last block. These can be labelled $R_1, R_2, \ldots, R_{p+1}$. In figure 1, we have the case $p = 3$. The row lengths for the Young Diagram R_1 are all positive, since negative rows for it would imply negative row lengths for the large chiral Young Diagram, which are not allowed for SU(N). The row lengths for $R_2, \ldots, R_{p+1}$ are allowed to be negative, subject to the usual restrictions $r_{i+1} \leq r_i$. Negative row lengths imply that the large chiral Diagram has a few boxes chipped away from the blocks of sizes $(k_1, Nl_1) \cdots (k_p, Nl_p)$. An example for $p = 1$ is shown in figure 3. Since the row lengths of $R_2, \ldots, R_{p+1}$ are small compared to N, these negative row lengths only make small modifications to the large blocks.

The basic formula for the Casimir of SU(N) is

$$C_2 = Nn + \sum_i r_i(r_i - 2i + 1) - \frac{n^2}{N}\,, \tag{2.1}$$

where r_i are row lengths and n is the total number of boxes in the Young Diagram.

This large chiral Young diagrams and their Casimirs can be described with the following notation. Define

$$\begin{aligned}
\hat{k}_a &= k_a - k_{a+1} \qquad \text{for} \qquad a = 1, \ldots, p-1 \\
\hat{k}_p &= k_p \\
\hat{l}_a &= l_1 + l_2 + \cdots + l_a \\
n_0 &= \sum_{a=1}^{p} \hat{k}_a \hat{l}_a = \sum_{a=1}^{p} k_a l_a \\
n_1 &= \sum_{a=1}^{p+1} n(R_a)\,.
\end{aligned} \tag{2.2}$$

So we have described a Young Diagram made out of large blocks described by $(\vec{k}, \vec{l})$ and small Young Diagrams $R_1, R_2, \ldots, R_{p+1}$. The row lengths for this large chiral Young Diagram are given by

$$\begin{aligned}
\text{For } i &= 1, \ldots, k_p \\
r_i(R) &= N\hat{l}_p + r_i(R_{p+1}) \\
\text{For } i &= (k_p + 1), \ldots, k_{p-1} \\
r_i(R) &= N\hat{l}_{p-1} + r_{i-k_p}(R_p) \\
&\ \ \vdots \\
\text{For } i &= (k_2 + 1), \ldots, k_1 \\
r_i(R) &= N\hat{l}_1 + r_{i-k_2}(R_2) \\
\text{For } i &= (k_1 + 1), \ldots, N \\
r_i(R) &= r_{i-k_1}(R_1)\,.
\end{aligned} \tag{2.3}$$

A more compact way to write this

$$\begin{aligned}
&\text{For} \quad a = p, \dots, 1 \qquad \text{for} \quad i = (k_{a+1}+1), \dots, k_a \\
&r_i(R) = N\hat{l}_a + r_{i-k_{a+1}}(R_{a+1}) \\
&\text{For} \quad i = (k_1+1), \dots, N \\
&r_i(R) = r_{i-k_1}(R_1)\,.
\end{aligned} \tag{2.4}$$

Note that, since R_1 is a small Young diagram with number of boxes small compared to N, the $r_i(R)$ given in the last line of (2.3), (2.4) vanish well before i reaches N.

Now the Casimir normalized by N is described by

$$\begin{aligned}
\frac{C_2}{N} &= N\left(\sum_{a=1}^{p} \hat{k}_a \hat{l}_a(\hat{l}_a+1)\right) + \\
&\quad + \left(n_1 - n_0^2 - \sum_{a=1}^{p} l_a k_a^2 + 2\sum_{a=1}^{p} \hat{l}_a n(R_{a+1})\right) + \\
&\quad + \frac{1}{N}\left(n_1 - 2n_0 n_1 + \sum_{a=1}^{p+1}\sum_{i=1}^{\hat{k}_{a-1}} r_i^2(R_a) - 2\sum_{a=1}^{p} k_a n(R_a) - \sum_{a=1}^{p}\sum_{i=1}^{\hat{k}_{a-1}} 2i r_i(R_a)\right) - \\
&\quad - \frac{1}{N^2}\sum_{a=1}^{p+1}(n^2(R_a)) \\
&= N c_0 + c_1 + \frac{c_2}{N} + \frac{c_3}{N^2}
\end{aligned} \tag{2.5}$$

In the last line we have defined $c_0, \dots, c_3$ which are to be read off from the previous equation. The term $\sum_{a=1}^{p} l_a k_a^2$ can also be written as $\sum_{a=1}^{p} \hat{l}_a (k_a^2 - k_{a+1}^2)$ where $k_{p+1} = 0$.

Redefining $\frac{g_{YM}^2 NA}{2} \to A$, the exact YM2 partition function (1.1) can be written as

$$Z = \sum_R e^{-AC_2/N}\,. \tag{2.6}$$

Using the above decomposition of the Casimir for choices $k_1, \dots, k_p$ and $l_1, \dots, l_p$ we can write

$$Z = Z_{\text{pert}} + \sum_{p=1}^{\infty} \sum_{k_1,\dots,k_p=1}^{\infty} \sum_{l_1,\dots,l_p=1}^{\infty} \sum_{R_1,\dots,R_{p+1}} e^{-ANc_0 - Ac_1 - Ac_2/N - Ac_3/N^2}\,, \tag{2.7}$$

where the part perturbative in the $1/N$ expansion is the chiral Gross-Taylor expansion.

In the above we have built a "large chiral Young Diagram" with the data

$$[k_1, k_2, \dots, k_p, l_1, l_2, \dots, l_p; R_1, \dots, R_{p+1}]\,,$$

where $k_1, \dots, k_p$ and $l_1, \dots l_p$ are positive integers and $R_1, \dots, R_{p+1}$ are Young Diagrams. To be brief, we denote them as $[\vec{k}, \vec{l}; \vec{R}]$. These Young diagrams have complex conjugates. The composite Young diagrams built by putting together diagrams of the above type with conjugates can be denoted as

$$\left[k_1, k_2, \dots, k_p, l_1, l_2, \dots, l_p; R_1, \dots, R_{p+1} \mid \bar{k}_1, \dots, \bar{k}_q, \bar{l}_1, \dots, \bar{l}_q; S_1, \dots, S_{q+1}\right].$$

More briefly we can denote them by $[\vec{k}, \vec{l}; \vec{R} \mid \vec{\bar{k}}, \vec{\bar{l}}; \vec{S}]$. We then need to compute the Casimirs of these non-chiral composites. We will do this in the next section.

3. e^{-AN} effects: the non-chiral diagrams

We consider the non-chiral Young Diagrams obtained by fusing a chiral Young Diagram and an anti-chiral Young Diagram. The chiral Young Diagram is specified by a pair of p-vectors of integers $\vec{k}, \vec{l}$ which specify rectangles of size $k_1 l_1 N, k_2 l_2 N, \ldots$ and a set of $p+1$ Young diagrams R_a of size order 1 as N goes to infinity. The k's are ordered $k_1 > k_2 > \cdots > k_p$. The antichiral diagram is specified by $\bar{k}_1, \ldots, \bar{k}_q$, $\bar{l}_1, \ldots, \bar{l}_q$ and Young Diagrams $S_1, \ldots, S_{q+1}$. The $\bar{k}$'s are ordered $\bar{k}_1 > \bar{k}_2 > \cdots > \bar{k}_q$. Figure 2 shows a non-chiral Young Diagram for the case $p = 3, q = 3$. Generalizing the possibility of negative row lengths in the chiral case, row-lengths for $S_2, \ldots, S_{q+1}$ can be negative.

The row lengths r_i of the composite Young Diagram R are as follows :

$$\begin{aligned}
&\text{For} \quad i = 1, \ldots, k_p \\
r_i(R) &= N(\widehat{\bar{l}_q} + \hat{l}_p) + r_1(S_{q+1}) + r_i(R_{p+1}) \\
&\text{For} \quad i = (k_p + 1), \ldots, k_{p-1} \\
r_i(R) &= N(\widehat{\bar{l}_q} + \hat{l}_{p-1}) + r_1(S_{q+1}) + r_{i-k_p}(R_p) \\
&\quad \vdots \\
&\text{For} \quad i = (k_2 + 1), \ldots, k_1 \\
r_i(R) &= N(\widehat{\bar{l}_q} + l_1) + r_1(S_{q+1}) + r_{i-k_2}(R_2) \\
&\text{For} \quad i = (k_1 + 1), \ldots, N - \bar{k}_1 \\
r_i(R) &= N\widehat{\bar{l}_q} + r_1(S_{q+1}) + r_{i-k_1}(R_1) - r_{N-\bar{k}_1-i+1}(S_1) \\
&\text{For} \quad i = (N - \bar{k}_1 + 1), \ldots, (N - \bar{k}_2) \\
r_i(R) &= N(\widehat{\bar{l}_q} - \widehat{\bar{l}_1}) + r_1(S_{q+1}) - r_{N-\bar{k}_2-i+1}(S_2) \\
&\quad \vdots \\
&\text{For} \quad i = (N - \bar{k}_q + 1), \ldots, N \\
r_i(R) &= r_1(S_{q+1}) - r_{N-i+1}(S_{q+1}) \,.
\end{aligned} \tag{3.1}$$

A more compact way to write this

$$\begin{aligned}
&\text{For} \quad a = p, \ldots, 1 \qquad \text{for} \;\; i = (k_{a+1} + 1), \ldots, k_a \\
r_i(R) &= N(\hat{\bar{l}}_q + \hat{l}_a) + r_1(S_{q+1}) + r_{i-k_{a+1}}(R_{a+1}) \,. \\
&\text{For} \quad i = (k_1 + 1), \ldots, (N - \bar{k}_1) \\
r_i(R) &= N\widehat{\bar{l}_q} + r_1(S_{q+1}) + r_{i-k_1}(R_1) - r_{N-\bar{k}_1-i+1}(S_1) \,. \\
&\text{For} \quad a = 1, \ldots, q \qquad \text{for} \;\; i = (N - \bar{k}_a + 1), \ldots, (N - \bar{k}_{a+1}) \\
r_i(R) &= N(\widehat{\bar{l}_q} - \widehat{\bar{l}_a}) + r_1(S_{q+1}) - r_{N-\bar{k}_{a+1}-i+1}(S_{a+1}) \,.
\end{aligned} \tag{3.2}$$

Although we are only using p-pairs of integers for the chiral Young Diagram $k_1, \ldots, k_p$; $l_1, \ldots, l_p$ and q pairs $\bar{k}_1, \ldots, \bar{k}_q; \bar{l}_1, \ldots, \bar{l}_q$ for the anti-chiral Young Diagram, it is useful to define $k_{p+1} = \bar{k}_{q+1} = 0$ in (3.2).

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Using these values for the row lengths we can compute the Casimir using (2.1) and we find

$$
\begin{aligned}
\frac{C_2}{N} = {} & N\left(\sum_{a=1}^{p}\hat{k}_a\hat{l}_a(\hat{l}_a+1)\right)+\sum_{a=1}^{q}\hat{\bar{k}}_a\hat{\bar{l}}_a(\hat{\bar{l}}_a+1)\quad)- \\
& -(n_0-\bar{n}_0)^2-\sum_{a=1}^{q}\bar{l}_a\bar{k}_a^2-\sum_{a=1}^{p}l_ak_a^2+ \\
& +\sum_{a=1}^{p+1}n(R_a)+\sum_{a=1}^{q+1}n(S_a)+2\sum_{a=1}^{p}\hat{l}_an(R_{a+1})+2\sum_{a=1}^{q}\hat{\bar{l}}_an(S_{a+1})+ \\
& +\frac{1}{N}\left(\sum_{a=1}^{p+1}n(R_a)+\sum_{a=1}^{q+1}n(S_a)\right)- \\
& -\frac{2}{N}\left(\sum_{a=1}^{p}k_an(R_a)+\sum_{a=1}^{q}\bar{k}_an(S_a)\right)- \\
& -\frac{2}{N}(n_0-\bar{n}_0)\left(\sum_{a=1}^{p+1}n(R_a)+\sum_{a=1}^{q+1}n(S_a)\right)+\frac{1}{N}\sum_{a=1}^{p+1}\sum_i r_i^2(R_a)+ \\
& +\frac{1}{N}\sum_{a=1}^{q+1}\sum_i r_i^2(S_a)-\frac{1}{N}\sum_{a=1}^{p+1}\sum_i 2ir_i(R_a)-\frac{1}{N}\sum_{a=1}^{q+1}\sum_i 2ir_i(S_a)- \\
& -\frac{1}{N^2}\left(\sum_{a=1}^{p+1}n(R_a)-\sum_{a=1}^{q+1}n(S_a)\right)^2 .
\end{aligned}
\tag{3.3}
$$

We have defined anti-chiral analogs of (2.2)

$$
\begin{aligned}
\hat{\bar{l}}_a &= \bar{l}_1+\bar{l}_2+\cdots+\bar{l}_a \\
\hat{\bar{k}}_a &= \bar{k}_a-\bar{k}_{a+1} \\
\bar{n}_0 &= \sum_{a=1}^{q}\bar{k}_a\bar{l}_a=\sum_{a=1}^{q}\hat{\bar{k}}_a\hat{\bar{l}}_a .
\end{aligned}
\tag{3.4}
$$

Note that this formula is symmetric under exchange of chiral and anti-chiral, which exchanges $p\leftrightarrow q$, $R_a\leftrightarrow S_a$. It also agrees with the chiral formula (3.3) when we set to zero the anti-chiral variables, $r_i(S_a), n(S_a), \bar{k}_a, \bar{l}_a, \bar{n}_0$.

From (3.3) we can read off $\frac{C_2}{N}=Nd_0+d_1+\frac{d_2}{N}+\frac{d_3}{N^2}$ and we get an expansion for the parition function (1.1) of the form.

$$
\begin{aligned}
Z = {} & Z_{\text{nchir}}^{\text{pert}}+\sum_{p=1}^{\infty}\sum_{k_1,\dots,k_p=1}^{\infty}\sum_{l_1,\dots,l_p=1}^{\infty}\sum_{R_1,\dots,R_{p+1}}e^{-ANc_0-Ac_1-Ac_2/N-Ac_3/N^2}+ \\
& +\sum_{q=1}^{\infty}\sum_{\bar{k}_1,\dots,\bar{k}_q=1}^{\infty}\sum_{\bar{l}_1,\dots,\bar{l}_q=1}^{\infty}\sum_{S_1,\dots,S_{q+1}}e^{-AN\bar{c}_0-A\bar{c}_1-A\bar{c}_2/N-A\bar{c}_3/N^2}+ \\
& +\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\sum_{k_1,\dots,k_p=1}^{\infty}\sum_{l_1,\dots,l_p=1}^{\infty}\sum_{\bar{k}_1,\dots,\bar{k}_q=1}^{\infty}\sum_{\bar{l}_1,\dots,\bar{l}_q=1}^{\infty}\sum_{R_1,\dots,R_{p+1}}\sum_{S_1,\dots,S_{p+1}}e^{-ANd_0-Ad_1-Ad_2/N-Ad_3/N^2} .
\end{aligned}
\tag{3.5}
$$

We have separated terms which are a series in $1/N$, the terms which involve e^{-AN} and are purely chiral (written out before in (2.5)), the terms which involve e^{-AN} and are purely anti-chiral, and finally terms involving e^{-AN} which are composites of chiral-antichiral. The constants $c_0, \ldots, c_3$ have been defined before and $\bar{c}_0, \ldots, \bar{c}_3$ are obtained by changing R_a to S_a , p to q , (k,l) to $(\bar{k},\bar{l})$. The sums over $k, \bar{k}$ are constrained by $k_1 > k_2 > \cdots > k_p$ and $\bar{k}_1 > \bar{k}_2 > \cdots \bar{k}_q$ as is clear from figures 1 and 2. We do not have a proof that the exponential contributions in (3.5) are complete. In fact one can write other types of diagrams which have $C_2 \sim N^2$, e.g. a Diagram having row lengths starting from $\sqrt{N}$ and dropping in steps of 1 down to zero, or a hook-shaped Young Diagram with first row of length $N/2$ and first column of length $N/2$. An important question is to find a complete description of the exponential corrections. It is also possible that the subset of terms picked out in (3.5) have a physical meaning in terms of D-branes in the equivalent topological string theory. D-branes will be discussed in further in section 5. Even if we have a complete description of the e^{-AN} terms, it is not clear that this would capture the full structure of the finite N partition function. For example we can certainly choose certain classes of Young Diagrams which contribute terms of the type e^{-AN^2}. It is worth noting that other scaling limits of 2dYM can be studied [13]. Whether the $1/N$ expansion and the exponential terms of the type e^{-AN} can allow us to reconstruct the finite N answer could be studied using techniques of exponential asymptotics [14].

4. Holomorphic/antiholomorphic mixing in SU(N) and U(N) partition functions

Define B as the number of boxes in the chiral Young Diagram and $\bar{B}$ as the number of boxes in the antichiral Young Diagram.

$$\begin{aligned}
B &= b_1 + b_0 \\
\bar{B} &= \bar{b}_1 + \bar{b}_0 \\
b_1 &= n_1 = \sum_{a=1}^{p+1} n(R_a) \\
b_0 &= N n_0 = N \sum_{a=1}^{p} k_a l_a \\
\bar{b}_1 &= \bar{n}_1 = \sum_{a=1}^{q+1} n(\bar{R}_a) \\
\bar{b}_0 &= N \bar{n}_0 = N \sum_{a=1}^{q} \bar{k}_a \bar{l}_a \,.
\end{aligned} \tag{4.1}$$

An important observation about the sum in (3.3) is that the mixed terms involving products of chiral and anti-chiral can be expressed entirely in terms of $(B - \bar{B})^2$. Indeed all such mixed terms in (3.3) are included in :

$$-(n_0 - \bar{n}_0)^2 - \frac{2}{N}(n_0 - \bar{n}_0)\left(\sum_{a=1}^{p+1} n(R_a) - \sum_{a=1}^{q+1} n(S_a)\right) - \frac{1}{N^2}(b_1 - \bar{b}_1)^2 \tag{4.2}$$

which is exactly equal to $-(B-\bar{B})^2$ when we use the fact

$$\begin{aligned} B &= b_0 + b_1 = N n_0 + \sum_a n(R_a) \\ \bar{B} &= \bar{b}_0 + \bar{b}_1 = N \bar{n}_0 + \sum_a n(S_a)\,. \end{aligned} \tag{4.3}$$

Defining[1]

$$Z_+ = \sum_{R_+} \exp\left(-\frac{1}{2} g^2_{\mathrm{YM}} C_2(R_+) + i\theta B\right)$$

we find that it can be written as

$$Z_+ = \sum_{R_+} \exp\left(-\frac{1}{2} g^2_{\mathrm{YM}} \kappa(R_+) - t|R_+|\right). \tag{4.4}$$

where $t = \frac{N g^2_{YM}}{2} - i\theta$ and we separated $C_2(R_+) = \kappa(R_+) + NB$. This can be viewed as a definition of $\kappa(R_+)$, and the explicit formula for it can be read off from (2.5)

$$\begin{aligned} \kappa(R_+) = {} & N \sum_{a=1}^{p} \hat{k}_a \hat{l}_a^2 - \\ & - n_0^2 + 2 \sum_{a=1}^{p} \hat{l}_a n(R_{a+1}) - \sum_{a=1}^{p} l_a k_a^2 + \\ & + \frac{1}{N} \left(n_1 - 2 n_0 n_1 + \sum_{a=1}^{p+1} \sum_{i=1}^{\hat{k}_{a-1}} r_i^2(R_a) - 2 \sum_{a=1}^{p} k_a n(R_a) - \sum_{a=1}^{p} \sum_{i=1}^{\hat{k}_{a-1}} 2 i r_i(R_a) \right) - \\ & - \frac{1}{N^2} \sum_{a=1}^{p+1} (n(R_a))^2\,. \end{aligned} \tag{4.5}$$

When we neglect the exponential corrections and keep only the $1/N$ expansion, $k_a = l_a = n_0 = 0$ and the only Young Diagram appearing is R_1. Then the formula for κ reduces, as expected, to $\sum_i r_i(R_1)(r_i(R_1) + 1 - 2i)$.

As we have observed above, all the mixed chiral-antichiral terms in (3.3) come from $(B-\bar{B})^2$. As a result, we can write

$$Z_{\mathrm{SU}(N)} = \exp\left[\frac{g^2_{YM}}{2N} (\partial_t - \partial_{\bar{t}})^2\right] Z_+(t) Z_-(\bar{t})\,. \tag{4.6}$$

The number of boxes in the chiral $\mathrm{SU}(N)$ Young Diagram is B , and the number of boxes in $\bar{S}$ (the conjugate of the anti-chiral Young Diagram S) is $N r_1(S) - \bar{B}$. Using the description of the Young Diagrams given before, $r_1(S) = r_1(S_{q+1}) + N \sum_{a=1}^{q} \bar{l}_a$. To the $\mathrm{SU}(N)$ Young Diagram we append a number r of columns of length N to the left. The $\mathrm{U}(1)$ charge is $q = Nr + N r_1(S) + B - \bar{B} = Nl + B - \bar{B}$, using the definition $l = r + r_1(S)$.

[1]Note that the partition function of 2dYM depends on $g^2_{YM} A, N$. In previous sections we had redefined $\frac{g^2_{YM} N A}{2} \to A$. Here we are setting $A = 1$ for comparison to [12].

To get the $\mathrm{U}(N)$ partition function, we use $C_2^{\mathrm{U}(N)} = C_2^{\mathrm{SU}(N)} + q^2/N$ and (1.1) to write

$$\begin{aligned} Z_{\mathrm{U}(N)} &= \exp\left(-\frac{g_{YM}^2 q^2}{2N}\right) Z_{\mathrm{SU}(N)} \\ &= \exp\left(-\frac{g_{YM}^2}{2N}(Nl + B - \bar{B})^2\right) Z_{\mathrm{SU}(N)} \,. \end{aligned} \tag{4.7}$$

As in the discussion of [12], the mixed term $\exp(-\frac{g_{YM}^2}{2N}(B-\bar{B})^2)$ cancels with the mixing term in the $\mathrm{SU}(N)$ partition function.

Then the same steps lead to

$$\tilde{Z}_{\mathrm{U}(N)} = \sum_{l=-\infty}^{\infty} \tilde{Z}_+(t + l g_{\mathrm{YM}}^2) \tilde{Z}_-(\bar{t} - l g_{YM}^2) \,. \tag{4.8}$$

Where $\tilde{Z}_+(t) = Z_+(t) e^{-\frac{t^3}{6 g_{\mathrm{YM}}^4} + \frac{t}{24}}$ and $\tilde{Z}_{\mathrm{U}(N)} = e^{\frac{-(t^3+\bar{t}^3)}{6 g_{\mathrm{YM}}^2} + \frac{t+\bar{t}}{24}} Z_{\mathrm{U}(N)}$. This is the same form as in [12], but the Z_+ and $\kappa(R_+)$ entering this formula are different and are given in (4.4) and (4.5).

5. D-branes and e^{-AN}

5.1 5-brane and Myers effect?

The $l = 1$ contributions to the partition function containing factors of e^{-AkN} were interpreted in terms of D-strings wrapped on the torus with winding number k. Now that we have found the extra parameter l in the space of Young diagrams contributing e^{-AN} effects and are getting $e^{-Akl(l+1)N}$, we can ask what is the interpretation of the factor $kl(l+1)$. Note that the discussion in [12] is in a IIA superstring theory context. For convenience the current discussion is in a IIB context. But the current discussion could be translated to a IIA set-up by considering 2-branes instead of 1-branes (as well 4 and 6-branes rather than 3 and 5-branes in the following) and it may also be interesting to study the effects of T-duality relating IIA and IIB.

One possibility is that the factor of $l(l+1)N = l(l+1)/g_s$ should be interpretd as a modified tension of D-strings. In the proposal of [12] for the string theory of 2dYM there are elementary strings on a CY which is a 4D space fibered over the T^2 of the 2dYM theory. We can get D-strings with modified tension, if the D-strings are actually extended objects in the extra directions. For example they could be 5-branes which wrap the T^2 as D-strings do, but extend into the extra directions. What kind of transverse geometry can explain the form $l(l+1)$?

The form $l(l+1)$ can be understood if we make an assumption about the geometry of these D-strings in the directions transverse to the 2-torus base. We assume that they are not points in the transverse dimensions but rather form a fuzzy geometry by the Myers dielectric effect [15]. It is a familiar fact in fuzzy geometry that when a D-string expands into a 5-brane to form a funnel with fuzzy 4-sphere cross-section, the number of D-strings involved cannot be an arbitrary integer but rather is constrained to be integers of the form $\frac{(n+1)(n+2)(n+3)}{6}$ where n is a positive integer [16 – 18]. If the transverse geometry is a fuzzy CP^2 [19, 20] then the allowed numbers of D-strings are dimensions of symmetric representations of SU(3) which, for a representation with p boxes is $\frac{(p+1)(p+2)}{2}$. Identifying $(p+1)$ with l above, we have a picture where $l(l+1)$ D-strings bound to form a fuzzy CP^2 wrap the T^2 with degree k.

Given that the extra dimensions of the Calabi-Yau are just given by fibres of line bundles $L_m \oplus L_{-m}$, there is no obvious CP^2 in the picture. The topological string of the CY is of course embedded in the physical string theory, which has an extra R^4 transverse to the CY. The CP^2 might involve directions internal to the CY, as well as directions transverse to the CY. It might also require taking a near horizon geometry of branes wrapping a 4-cycle on the CY, since the gauge theory on such branes is what gives rise to the 2dYM in the picture of [12].

5.2 Other possibilities: 3-brane?

Another possibility is that the transverse geometry of the branes rsponsible for the e^{-AN} effects is two-dimensional. So the D-strings are secretly a D3-branes. Two of their world-volume directions wrap around the T^2 base of the Calabi-Yau and the remaining two directions wrap directions internal to the fibre over the T^2. If the winding number in the internal directions is restricted to be of the form $l(l+1)$, then from the point of view of the T^2 base, the $D3$-brane looks like a $D1$ string with effective tension proportional to $l(l+1)$. It is not clear what mechanism would restrict the winding numbers to be of the above form. It is possible that we have to use the K-theory approach to D-brane charges [21, 22] in order to understand the restricted form of the winding numbers.

5.3 Stringy excitations of D-brane systems

The detailed structure of (2.5) and (3.3) should contain enough information to distinguish between the possibilities above. A given term going like e^{-AN} is multiplied by a series $P(1/N)$ which should be accounted by strings stretching between the D-branes. A lot of the terms in (3.5) are directly related to the quadratic Casimirs of the Young Diagrams $R_1, \ldots, R_{p+1}$ and $S_1, \ldots, S_{q+1}$. In the case where we restrict attention to perturbative $(1/N)$ terms, there is only R_1, S_1 which have been interpreted in terms of closed elementary strings [2, 3]. The full set $R_1, \ldots, R_{p+1}$ should be related to closed as well as open elementary strings, stretching between different D-branes, where the D-branes are labelled by $(\hat{k}_a, \hat{l}_a)$. Since R_1 does not appear in second line of (2.5) it is distinguished from the $R_2, \ldots, R_{p+1}$. We have also noted that the latter can have negative row lengths. The negative row lengths correspond to a left-right and up-down reflected Young Diagram. The positive part of $R_2, \ldots, R_{p+1}$ can be associated with open strings starting and ending on the p D-branes and mapping holomorphically to the target. The negative parts can be associated to similar open strings mapping anti-holomorphically to the target. The presence of p sectors of such open string fluctuations suggests the D-branes are in phase where U(p) is broken to U$(1)^p$. There may also be a closed string description of the excitations around the D-brane.

The integers $(\bar{k}, \bar{l})$ characterizing the anti-chiral parts of the composite Young Diagrams would correspond to anti-branes. $S_2, \ldots, S_{q+1}$ would be related to stringy excitations of the q anti-D-branes. The positive and negative parts would map to T^2 holomorphically and anti-holomorphically respectively. The precise nature of the D-branes will affect the form of the coupling between the open strings and the D-branes. Hence the D5 versus D3 possibility could be resolved. We leave this for future work.

Another interesting avenue in connection with the spectrum of branes that might be relevant to an understanding of the exponential type terms in 2dYM involves fractional D-strings. While all the terms considered so far (and in the bulk of the paper) involve integral powers of e^{-AN}, we can also get fractional powers. The large blocks we considered so far have lengths lN, where l's are integers. If we allow these to be fractional, the terms of type $e^{-ANl(l+1)}$ will give fractional powers of e^{-AN}. For example if $l = \frac{1}{m}$ with m large compared to 1 but small compared to N we can get $e^{-AN/m}$. These appear to be contributions from fractional D-strings. Fractional branes have been discussed before in [23 − 25].

6. Conclusions

We have computed a class of non-perturbative contributions to the partition function for two-dimensional Yang Mills theory on a torus, behaving like powers of e^{-AN} multplied by series in $1/N$. These contributions come from a class of large Young Diagrams, generalizing the ones that were introduced in this context in [8]. It will be interesting to investigate whether the exponential terms we have described are in an appropriate sense complete e.g whether they provide the complete description of D-branes in the dual topological string theory. The free fermion description of 2dYM may be useful in exploring this question.

We showed that structure of the mixing between chiral and anti-chiral sectors, for the SU(N) and the U(N) partition functions, is very similar to the one described in [12] for the $1/N$ expansion. We discussed the geometrical interpretation of the non-perturbative contributions in terms of D-branes. This discussion is far from complete, but it suggests that a proper geometrical understanding requires the use of more than the T^2 base of the two-dimensional Yang-Mills and should involve the Calabi-Yau described in [12] or the full ten-dimensional background of a physical string related to the topological string of the CY. One of the approaches we discussed for explaining the detailed form of the exponential contributions involves D-strings blowing up into a D5-branes having a transverse fuzzy CP^2 geometry. Further clarifying the D-brane picture is a very interesting direction for the future.

It will be interesting to explore the meaning of the non-perturbative holomorphic-antiholomorphic mixing in the light of connections to black hole entropy [12, 11, 26, 27]. Large Young diagrams have also appeared recently in the dual gauge theory description of non-perturbative objects in ADS/CFT [28 − 33]. Large horizontal Young diagrams in that context are related to ADS-giant gravitons. Whether there is a clear connection between the geometry of large Young Diagrams in ADS/CFT and in two-dimensional Yang Mills is an interesting question.

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After this paper was completed, an interesting work on non-perturbative structure of two-dimensional Yang-Mills has appeared [34]. We have focused on contributions to the partition function which involve, in the free fermion picture, a few fermions (order one in number) near the top of the fermi sea excited by steps comparable to N, and their duals where fermions near the bottom of the fermi sea are excited. Excitations breaking the simple mixing of chiral-antichiral involving excitations from deep inside the fermi sea were analyzed in [34]. Deriving exlicit formulae for the exponential terms in two-dimensional Yang Mills will require incorporating both kinds of excitations and is left for the future.

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